§3a SL_2 / Virasoro case following [Maulik–Okounkov]

$$\begin{array}{ll} \tilde{\mathcal{U}}_2^d & : \text{Gieseker space for rk 2 torsion free sheaves} \\ \downarrow & \\ \mathcal{U}^d_{SL_2} & \curvearrowleft \ \mathbb{T} = \underset{\varepsilon_1,\varepsilon_2}{T^2} \times \underset{\substack{a_1,a_2 \\ a_1+a_2=0}}{T} \subset GL_2 \times SL_2 \end{array}$$

Th [MO, Schiffmann–Vasserot]

$$\bigoplus_d H^*_{\mathbb{T},c}(\tilde{\mathcal{U}}_2^d) \subset \bigoplus_d H^*_{\mathbb{T}}(\tilde{\mathcal{U}}_2^d) : \text{rep. of } W_A(\mathfrak{gl}_2) = Vir_A \otimes Heis_A(\mathbb{C})$$

↑ Verma (universal)

↑ dual Verma module with highest weight $= \vec{a}$ } explained later

$$H^0_{\mathbb{T}}(\tilde{\mathcal{U}}_2^0) = H^0_{\mathbb{T}}(pt) \ni |1^0\rangle : \text{highest weight vector}$$

$G = SL_2$

$$\begin{tikzcd}
 & \mathcal{U}^d_G \arrow[r, phantom, "\overset{j}{\supset}"] & \mathcal{U}^d_B \arrow[r, shift right, "p"'] & \mathcal{U}^d_T \arrow[l, shift right, "i"'] \arrow[r, phantom, "="] & S^d\mathbb{C}^2 & \coprod S^{d_1}\mathbb{C}^2 \times S^{d_2}\mathbb{C}^2 \arrow[l, "\text{sum}"'] \\
\text{Gieseker} & \tilde{\mathcal{U}}^d_2 \arrow[u] \arrow[r, phantom, "\overset{\tilde{j}}{\supset}"] & \tilde{\mathcal{U}}^d_B \arrow[u] \arrow[r, phantom, "\overset{\tilde{i}}{\supset}"] \arrow[r, dashed, bend right, "\cancel{\tilde{p}}"'] & \tilde{\mathcal{U}}^d_T \arrow[u] \arrow[r, phantom, "="] & \coprod_{d_1+d_2=d} X^{[d_1]} \times X^{[d_2]} \arrow[ur] &
\end{tikzcd}$$

$$\tilde{\mathcal{U}}^d_2 = \{E \mid \text{torsion free}\}, \quad \tilde{\mathcal{U}}^d_B = \{E = E_0 \supset E_1 \supset E_2 = 0\}, \quad \tilde{\mathcal{U}}^d_T = \{E = I_1 \oplus I_2\}$$

$$\bigoplus_d H^*_{\mathbb{T},c}(\tilde{\mathcal{U}}^d_2) = \bigoplus H^*_{\mathbb{T},c}(X^{[d_1]}) \otimes H^*_{\mathbb{T},c}(X^{[d_2]}) \cong \text{Fock module for } Heis_A(\mathfrak{h}_{\mathfrak{gl}_2})$$

$$W_{\mathbb{A}}(\mathfrak{gl}_2) = W_{\mathbb{A}}(\mathfrak{sl}_2) \otimes \mathrm{Heis}_{\mathbb{A}}(\mathfrak{g}_{diag.}) \subset \mathrm{Heis}_{\mathbb{A}}(\mathfrak{g}_{\substack{anti\\diag.}}) \otimes \mathrm{Heis}_{\mathbb{A}}(\mathfrak{g}_{diag.}) = \mathrm{Heis}_{\mathbb{A}}(\mathfrak{g}_{\mathfrak{gl}_2})$$

Therefore it is enough to relate $H^*_{\mathbb{T},c}(\tilde{\mathcal{U}}^d_Z)$ and $H^*_{\mathbb{T},c}(\tilde{\mathcal{U}}^d_T)$

A naive guess $H^*_{\mathbb{T},c}(\tilde{\mathcal{U}}^d_r) \xrightarrow[\text{pull back}]{} H^*_{\mathbb{T},c}(\tilde{\mathcal{U}}^d_T)$ does not work.

[MO] stable envelop

$$\tilde{\mathcal{U}}^d_r \times \tilde{\mathcal{U}}^d_T \supset Z = \tilde{\mathcal{U}}^d_B \times_{\mathcal{U}^d_T} \tilde{\mathcal{U}}^d_T \quad : \text{lagrangian}$$

[MO] constructed a certain lag. cycle $\mathcal{L} \in H_{top}(Z)$ (stable envelop)

s.t. $Z \overset{I}{\hookleftarrow} \tilde{\mathcal{U}}^d_T \times_{\mathcal{U}^d_T} \tilde{\mathcal{U}}^d_T \qquad I^*[\mathcal{L}] = \pm e(N_+ \otimes 1) \cap [\Delta \tilde{\mathcal{U}}^d_T] \cancel{+ \cdots}$

no error term

Then they define $\mathrm{Stab}_B : H^*_{\mathbb{T},c}(\tilde{\mathcal{U}}^d_r) \longrightarrow H^*_{\mathbb{T},c}(\tilde{\mathcal{U}}^d_T)$

$$p_{2*}(p_1^*(\cdot) \cap [\mathcal{L}])$$

This depends on the choice of $T \subset B \subset SL_2$ $\qquad \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ or $\begin{bmatrix} * & 0 \\ * & * \end{bmatrix}$

$\mathrm{Stab}_{\mathfrak{B}}$ is an isomorphism over $\otimes_{H^*_{\mathbb{T}}(pt)} \mathrm{Frac}(\ \ )$

$$\begin{tikzcd}
H^*_{\mathbb{T},c}(\widetilde{\mathcal{U}}^d_2) \otimes \mathrm{Frac} \arrow[r, "\mathrm{Stab}_{\mathfrak{B}}"] \arrow[dr, "\mathrm{Stab}_{\mathfrak{B}'}"'] & H^*_{\mathbb{T},c}(\widetilde{\mathcal{U}}^d_T) \otimes \mathrm{Frac} \arrow[d] \\
& H^*_{\mathbb{T},c}(\widetilde{\mathcal{U}}^d_T) \otimes \mathrm{Frac}
\end{tikzcd}$$

$\mathrm{Stab}_{\mathfrak{B}'} \otimes \mathrm{Stab}_{\mathfrak{B}}^{-1} =: R$ instanton R-matrix

algebraic terms: antidiag diag.

$$\begin{array}{ccccc}
\mathrm{Fock}(a_1) \otimes \mathrm{Fock}(a_2) & = & \mathrm{Fock}(a_1 - a_2) \otimes \mathrm{Fock}(a_1 + a_2) & \hookleftarrow & W_{\mathbb{A}}(\mathfrak{gl}_2) \\
b_0^{(1)} = a_1 \mathrm{id} \quad R \downarrow \quad b_0^{(2)} = a_2 \mathrm{id} & & \downarrow R^{\mathfrak{sl}_2} \otimes \mathrm{id} & \swarrow & \\
\mathrm{Fock}(a_2) \otimes \mathrm{Fock}(a_1) & = & \mathrm{Fock}(a_2 - a_1) \otimes \mathrm{Fock}(a_1 + a_2) & &
\end{array}$$

$R^{\mathfrak{sl}_2}$ intertwines two embeddings of $W_{\mathbb{A}}(\mathfrak{gl}_2)$.

L_0 acts on $|1^0\rangle$ by $(a_1 - a_2)^2 + c(\varepsilon_1, \varepsilon_2)$ $\therefore$ invariant under $a_1 \leftrightarrow a_2$

generic h.w. $\Rightarrow$ Fock sp. is irreducible as Vir-module

$\therefore \ \exists 1$ intertwiner $\mathrm{Fock}(a_1 - a_2) \longrightarrow \mathrm{Fock}(a_2 - a_1)$

Th ([MO])

instanton R-matrix $=$ intertwiner

$\Longrightarrow W_{A}(gl_2) \curvearrowright \bigoplus_d H^*_{T,c}(\widetilde{\mathcal{U}}^d_2)$

Rem $P_m(\alpha)$ and $c_i(\mathcal{V})$ are defined exactly as in $X^{[n]}$.
They generate $W_A(gl_2)$

☆ Yang-Baxter equation

$$R_{12}(a_1-a_2)R_{13}(a_1-a_3)R_{23}(a_2-a_3) = R_{23}(a_2-a_3)R_{13}(a_1-a_3)R_{12}(a_1-a_2)$$

geometric side : $H^*_{T,c}(\widetilde{\mathcal{U}}^d_3)$

algebraic side : $\mathrm{Fock}(a_1) \otimes \mathrm{Fock}(a_2) \otimes \mathrm{Fock}(a_3)$

Both LHS, RHS are intertwiners of

$\mathrm{Vir}_{1,A} \otimes \mathrm{Heis}(\alpha_1^\perp) \cap \mathrm{Vir}_{2,A} \otimes \mathrm{Heis}\,\alpha_2^\perp = W_{A_2}(gl_3)$

$\exists 1 \Rightarrow$ LHS $=$ RHS //

§3b.

main result

Assume G : ADE

$\mathcal{U}_G^d \curvearrowleft T^2 \times T = \mathbb{T} \subset GL_2 \times G$ $\qquad H^*_{\mathbb{T}}(pt) = \mathbb{C}[\varepsilon_1, \varepsilon_2, \vec{a}]$

(1) $\underbrace{\bigoplus_d IH^*_{\mathbb{T},c}(\mathcal{U}_G^d)}_{= M_{\vec{a}}} \subset \overbrace{\bigoplus_d IH^*_{\mathbb{T}}(\mathcal{U}_G^d)}^{= M_{\vec{a}}^\vee}$ are representations of $W_{\vec{a}}(\mathfrak{g})$

Verma module $\qquad$ dual Verma module with highest weight $\vec{a}$

$IH^0_{\mathbb{T},c}(\mathcal{U}_G^0) = IH^0_{\mathbb{T},c}(pt) \ni |1^0\rangle$: highest weight vector

(2) $1^d \in IH^*_{\mathbb{T}}(\mathcal{U}_G^d)$ (fundamental class) satisfies the Whittaker vector condition :

$m>0 \qquad \tilde{W}_m^{(i)} 1^d = \begin{cases} \pm 1^{d-1} & \text{if } m=1,\ i=\ell \text{ (top degree)} \\ 0 & \text{otherwise} \end{cases}$

suitably normalised generator

$G = SL_r$ $\qquad$ Maulik-Okounkov , Schiffmann-Vasserot

$G = SL_r$ $\quad$ $\tilde{\mathcal{U}}_r^d$: Gieseker space $\quad$ $\pi : \tilde{\mathcal{U}}_r^d \to \mathcal{U}^d_{SL_r} = \coprod \mathrm{Bun}^{d'}_{SL_r} \times S_\lambda \mathbb{C}^2$

$$\Rightarrow H^*_{\mathbb{T}}(\tilde{\mathcal{U}}_r^d) = \bigoplus_{d',\lambda} IH^*_{\mathbb{T}}(\mathcal{U}^{d'}_{SL_r} \times S_\lambda \mathbb{C}^2) \otimes H_{top}(\pi^{-1}(x_{d',\lambda}))$$

1 dim'l $\qquad$ $x_{d',\lambda} \in \mathrm{Bun}^{d'}_{SL_r} \times S_\lambda \mathbb{C}^2$

$$\Rightarrow \bigoplus_d H^*_{\mathbb{T}}(\tilde{\mathcal{U}}_r^d) \cong \Big(\bigoplus_d IH^*_{\mathbb{T}}(\mathcal{U}^d_{SL_r})\Big) \otimes \Big(\bigoplus_\lambda \mathbb{C}\Big)$$

$\curvearrowleft W_{\mathbb{A}}(\mathfrak{gl}_r)$ $\qquad$ $\curvearrowleft W_{\mathbb{A}}(\mathfrak{sl}_r)$ $\qquad$ $\curvearrowleft$ Fock for $\mathrm{Heis}(\mathbb{C})$

☆ Wakimoto module

We construct modules of $\mathrm{Heis}_{\mathbb{A}}(\mathfrak{h})$ $\quad$ $M_{\mathbb{A}} \subset N_{\mathbb{A}} \subset N^\vee_{\mathbb{A}} \subset M^\vee_{\mathbb{A}}$

$$\mathcal{U}^d_G \xleftarrow{j} \mathcal{U}^d_P \underset{i}{\overset{p}{\rightleftarrows}} \mathcal{U}^d_T$$

We consider cohomology groups relatively as constructible sheaves

$\mathbb{C}_X$: constant sheaf

$$X \xrightarrow{p} pt$$

$$H^*(X) = H^*(p_* \mathbb{C}_X)$$

$$H^*_c(X) = H^*(p_! \mathbb{C}_X)$$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p_X}\searrow & \downarrow {\scriptstyle p_Y} \\ & & pt \end{array}$$

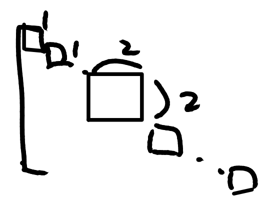
$H^*(Y) \xrightarrow{f^*} H^*(X)$ vs $p_{Y*}\mathbb{C}_Y \longrightarrow p_{Y*} f_* \underbrace{f^* \mathbb{C}_Y} = p_{X*} \mathbb{C}_X$

$\cdots$ etc

We consider Braden's hyperbolic restriction

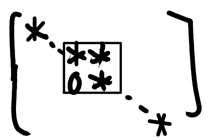
$$i^* j^! IC(\mathcal{U}^d_G) \in D^b(\mathcal{U}^d_T)$$

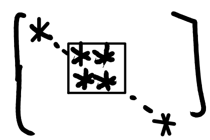
$$\begin{array}{ccc} H^*_c(i^! j^! IC(\mathcal{U}^d_G)) & \xrightarrow{\star} & H^*_c(i^* j^! IC(\mathcal{U}^d_G)) \overset{!!}{\longleftarrow} \text{Fock sp. for } Heis_A(\mathfrak{h}) \\ \wr\Vert & & \\ IH^*_{D,c}(\mathcal{U}^d_G) & & \end{array}$$



L_i ... Levi corr. to α_i $\qquad [L_i, L_i] \cong SL_2$

$$\begin{array}{ccccc} \mathcal{U}^d_B & \subset & \mathcal{U}^d_{P_i} & \subset & \mathcal{U}^d_G \\ \cup & & \cup & & \\ \mathcal{U}^d_{B\cap L_i} & \subset & \mathcal{U}^d_{L_i} & & \\ \cup & & & & \\ \mathcal{U}^d_T & & & & \end{array}$$





$$\begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix} \mathcal{U}^d_T$$

$\Longrightarrow$ $\star$ factors the Verma module of $Vir_{i,A} \ltimes Heis_A(\alpha_i^\perp)$

$\Longrightarrow$ (1) $\qquad$ In fact, we use L_i to prove !!